

# ON FINITELY PRESENTED ALGEBRAS

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**ABSTRACT.** We prove that if  $A$  is finitely presented algebra with idempotent  $e$  such that  $A = AeA = A(1 - e)A$  then the algebra  $eAe$  is finitely presented.

## 1. INTRODUCTION

Let  $F$  be a field and let  $A$  be an associative  $F$ -algebra generated by a finite collection of elements  $a_1, \dots, a_m$ .

Consider the free associative algebra  $F\langle x_1, \dots, x_m \rangle$  and the homomorphism  $F\langle x_1, \dots, x_m \rangle \xrightarrow{\varphi} A$ ,  $x_i \mapsto a_i$ .

We say that the algebra  $A$  is finitely presented (f.p.) if the ideal  $I = \ker \varphi$  is finitely generated as an ideal. This property does not depend on a choice of a generating system of  $A$  as long as this system is finite.

## 2. MAIN RESULT

**Theorem 1.** *Let  $A = A_{\bar{0}} + A_{\bar{1}}$  be a  $\mathbb{Z}/2\mathbb{Z}$ -graded associative algebra, such that  $A_{\bar{0}} = A_{\bar{1}}A_{\bar{1}}$ . If the algebra  $A$  is finitely presented then so is the algebra  $A_{\bar{0}}$ .*

Let  $A$  be a finitely generated associative algebra with an idempotent  $e$ .

S. Montgomery and L. Small [2] showed that if  $A = AeA$  then the Peirce component  $eAe$  is also finitely generated.

**Theorem 2.** *Let  $A$  be a finitely presented algebra with an idempotent  $e$  such that  $A = AeA = A(1 - e)A$ . Then the Peirce component  $eAe$  is finitely presented.*

*Proof of Theorem 1.* By our assumption the algebra  $A$  is generated by the subspace  $A_{\bar{1}}$ . Let  $a_1, \dots, a_m \in A_{\bar{1}}$  be the generators of  $A$ . Consider the free algebra  $F\langle X \rangle$ ,  $X = \{x_1, \dots, x_m\}$ , and the homomorphism  $f\langle X \rangle \xrightarrow{\varphi} A$ ,  $x_i \mapsto a_i$ . The algebra  $F\langle X \rangle$  is  $\mathbb{Z}/2\mathbb{Z}$ -graded,  $F\langle X \rangle = F\langle X \rangle_{\bar{0}} + F\langle X \rangle_{\bar{1}}$ , where  $F\langle X \rangle_{\bar{0}}$  (resp.  $F\langle X \rangle_{\bar{1}}$ ) is spanned by words in  $X$  of even (resp. odd) length. It is easy to

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see that the homomorphism  $\varphi$  is graded:  $\varphi(F\langle X \rangle_{\bar{0}}) = A_{\bar{0}}$ ,  $\varphi(F\langle X \rangle_{\bar{1}}) = A_{\bar{1}}$ . The subalgebra  $F\langle X \rangle_{\bar{0}}$  is freely generated by  $m^2$  elements  $x_i x_j$ ,  $1 \leq i, j \leq m$ . Let  $I = \ker \varphi$ ,  $I = I_{\bar{0}} + I_{\bar{1}}$ ,  $I_i = I \cap F\langle X \rangle_{\bar{i}}$ ,  $i = 0, 1$ . Our aim is to show that  $I_{\bar{0}}$  is a finitely generated ideal in  $F\langle X \rangle_{\bar{0}}$ .

Since the algebra  $A$  is finitely presented it follows that the ideal  $I$  is generated ( as an ideal ) by a finite set  $M = M_{\bar{0}} \cup M_{\bar{1}}$ ,  $M_i = F\langle X \rangle_{\bar{i}}$ ,  $i = 0, 1$ .

Consider the finite set

$$M' = \{a, x_i b, b x_i, x_i a x_j \mid a \in M_{\bar{0}}, b \in M_{\bar{1}}, 1 \leq i, j \leq m\}$$

We claim that  $I_{\bar{0}} = F\langle X \rangle_{\bar{0}} M' F\langle X \rangle_{\bar{0}}$ . Indeed, The equality  $I = F\langle X \rangle M F\langle X \rangle$  implies

$$I_{\bar{0}} = F\langle X \rangle_{\bar{0}} M_{\bar{0}} F\langle X \rangle_{\bar{0}} + F\langle X \rangle_{\bar{1}} M_{\bar{0}} F\langle X \rangle_{\bar{1}} + F\langle X \rangle_{\bar{0}} M_{\bar{1}} F\langle X \rangle_{\bar{1}} + F\langle X \rangle_{\bar{1}} M_{\bar{1}} F\langle X \rangle_{\bar{0}}.$$

Now,  $F\langle X \rangle_{\bar{1}} = F\langle X \rangle_{\bar{0}} X = X F\langle X \rangle_{\bar{0}}$ . Hence,

$$F\langle X \rangle_{\bar{1}} M_{\bar{0}} F\langle X \rangle_{\bar{1}} \subseteq F\langle X \rangle_{\bar{0}} X M_{\bar{0}} X F\langle X \rangle_{\bar{0}} \subseteq F\langle X \rangle_{\bar{0}} M' F\langle X \rangle_{\bar{0}};$$

$$F\langle X \rangle_{\bar{0}} M_{\bar{1}} F\langle X \rangle_{\bar{1}} \subseteq F\langle X \rangle_{\bar{0}} M_{\bar{1}} X F\langle X \rangle_{\bar{0}} \subseteq F\langle X \rangle_{\bar{0}} M' F\langle X \rangle_{\bar{0}};$$

$$F\langle X \rangle_{\bar{1}} M_{\bar{1}} F\langle X \rangle_{\bar{0}} \subseteq F\langle X \rangle_{\bar{0}} X M_{\bar{1}} F\langle X \rangle_{\bar{0}} \subseteq F\langle X \rangle_{\bar{0}} M' F\langle X \rangle_{\bar{0}}.$$

This proves the claim and finishes the proof of Theorem 1  $\square$

The reverse assertion is not true. There exists a finitely generated  $\mathbb{Z}/2\mathbb{Z}$ -graded associative algebra  $A = A_{\bar{0}} + A_{\bar{1}}$  such that  $A_{\bar{0}} = A_{\bar{1}} A_{\bar{1}}$ , the algebra  $A_{\bar{0}}$  is finitely presented, but the algebra  $A$  is not.

**Example 1.** Let  $A = \langle x, y \mid x^2 = yxy = 0, xy^{2i+1}x = 0 \text{ for } i \geq 1 \rangle$ , the elements  $x, y$  are odd. The subspace  $A_{\bar{0}}$  (resp.  $A_{\bar{1}}$ ) is spanned by the products of even ( resp. odd ) length in  $x, y$ . It is easy to see that the algebra  $A$  is not finitely presented. The subalgebra  $A_{\bar{0}}$  is generated by the elements  $a = xy, b = y^2, c = yx$  and presented by the relations  $cb = ca = ba = a^2 = c^2 = 0$ .

*Proof of Theorem 2.* Let  $A_{\bar{0}} = eAe + (1-e)A(1-e)$ ,  $A_{\bar{1}} = eA(1-e) + (1-e)Ae$ . Then  $A = A_{\bar{0}} + A_{\bar{1}}$  is a  $\mathbb{Z}/2\mathbb{Z}$ -grading of the algebra  $A$ .

By the assumptions of the theorem

$$eAe = eA(1-e)Ae \subseteq A_{\bar{1}}A_{\bar{1}}$$

and

$$(1-e)A(1-e) = (1-e)AeA(1-e) \subseteq A_{\bar{1}}A_{\bar{1}}.$$

By Theorem 1 the algebra  $A_{\overline{0}} = eAe \oplus (1 - e)A(1 - e)$  is finitely presented. It is well known [1] that a direct sum of two algebras is finitely presented if and only if both summands are finitely presented.

Hence the algebra  $eAe$  is finitely presented.  $\square$

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